ISRAEL JOURNAL OF MATHEMATICS **161** (2007), 17–27
DOI: 10.1007/s11856-007-0070-2

A NICHTNEGATIVSTELLENSATZ FOR POLYNOMIALS IN NONCOMMUTING VARIABLES*

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ABSTRACT

Let $S\cup\{f\}$ be a set of symmetric polynomials in noncommuting variables. If f satisfies a polynomial identity $\sum_i h_i^* f h_i = 1 + \sum_i g_i^* s_i g_i$ for some $s_i \in S \cup \{1\}$, then f is obviously nowhere negative semidefinite on the class of tuples of nonzero operators defined by the system of inequalities $s \geq 0$ $(s \in S)$. We prove the converse under the additional assumption that the quadratic module generated by S is Archimedean.

1. Introduction and main results

We write $\mathbb{N} := \{1, 2, \ldots\}$, $\mathbb{R}$ and $\mathbb{C}$ for the sets of natural, real and complex numbers. For $k \in \{\mathbb{R}, \mathbb{C}\}$, we consider the algebra $k\langle\bar{X}\rangle$ of polynomials in n noncommuting variables $\bar{X} := (X_1, \ldots, X_n)$ with coefficients from k. The elements of $k\langle\bar{X}\rangle$ are linear combinations of words in n letters from $\bar{X}$. The length of the

* Supported by the European network RAAG (EC contract HPRN-CT-2001-00271).

† Igor Klep was supported from the state budget by the Slovenian Research Agency (project No. Z1-9570-0101-06).
Received September 26, 2005

longest word in such a linear combination is called the **degree**. We equip $k\langle \bar{X} \rangle$ with the involution $*$ reversing the words, e.g., $(X_1X_2 + X_1^2)^* = X_2X_1 + X_1^2$. Thus $k\langle \bar{X} \rangle$ is the $*$-algebra freely generated by n symmetric elements. Let $\operatorname{Sym} k\langle \bar{X} \rangle$ denote the set of all symmetric elements, that is, $\operatorname{Sym} k\langle \bar{X} \rangle = \{f \in k\langle \bar{X} \rangle \mid f = f^*\}$. A polynomial of the form g^*g is called a **Hermitian square**.

If $f \in k\langle \bar{X} \rangle$ is a sum of Hermitian squares and we substitute bounded self-adjoint operators $A_1, \dots, A_n$ on the same Hilbert space for the variables $\bar{X}$, then the resulting operator $f(A_1, \dots, A_n)$ is positive semidefinite. For self-adjoint operators A and B on a Hilbert space E, we write $A \le B$ (respectively $A < B$) to express that $B - A$ is positive semidefinite (respectively positive definite), i.e.,

$$A \le B :\Leftrightarrow \langle Av, v \rangle \le \langle Bv, v \rangle \quad \text{for all } v \in E$$
$$A < B :\Leftrightarrow \langle Av, v \rangle < \langle Bv, v \rangle \quad \text{for all } v \in E \setminus \{0\}.$$

Helton [Hel] proved (a slight variant of) the converse of the above observation: If $f \in k\langle \bar{X} \rangle$ and $f(A_1, \dots, A_n) \ge 0$ for all self-adjoint matrices A_i of the same size, then f is a sum of Hermitian squares. For a beautiful exposition, we refer the reader to [MP].

We follow the terminology and notation used (in the commutative case) by Marshall in [Mar]. Fix a subset $S \subseteq \operatorname{Sym} k\langle \bar{X} \rangle$. The **semialgebraic 'set'** (called positivity domain in [HM]) K_S, associated to S, is the class of tuples $A = (A_1, \dots, A_n)$ of bounded self-adjoint operators on a *nontrivial*[1] k-Hilbert space making $s(A)$ a positive semidefinite operator for every $s \in S$. The **quadratic module** M_S is the set of all elements of the form $\sum_i g_i^* s_i g_i$, where $s_i \in S \cup \{1\}$ and $g_i \in k\langle \bar{X} \rangle$, i.e., the smallest subset of $\operatorname{Sym} k\langle \bar{X} \rangle$ satisfying $\{1\} \cup S \subseteq M_S$, $M_S + M_S \subseteq M_S$ and $g^* M_S g \subseteq M_S$ for all $g \in k\langle \bar{X} \rangle$. We call M_S **Archimedean** if there exists $N \in \mathbb{N}$ with $N - (X_1^2 + \cdots + X_n^2) \in M_S$. In this case, K_S is **bounded**; there exists $N \in \mathbb{N}$ such that $N - (A_1^2 + \ldots + A_n^2) \ge 0$ whenever $(A_1, \dots, A_n) \in K_S$. In other words, the operator norm is bounded uniformly for all operators appearing in a tuple belonging to K_S. As in the commutative case [JP], boundedness of K_S does not imply that M_S is Archimedean, see Example 1.1 below. Nevertheless, if K_S is bounded, we can add $N - (X_1^2 + \cdots + X_n^2)$ to S for a sufficiently big $N \in \mathbb{N}$ to make M_S Archimedean without changing K_S.

1 In contrast to [HM], we exclude the trivial Hilbert space $\{0\}$. This choice, of course, does not affect the validity of Theorem 1.2 below [HM, Theorem 1.2] but is necessary for Theorem 1.4 to hold.

1.1. Example: Let $n = 2$ and write $\bar{X} = (X, Y)$, i.e., we consider $k\langle\bar{X}\rangle = k\langle X, Y\rangle$. For

$$S := \{X - 1, Y - 1, 8 - XY - YX\},$$

K_S is bounded, but M_S is not Archimedean.

To see the latter, consider the k-algebra homomorphism $\pi\colon k\langle X, Y\rangle \to k[X, Y]$ making the variables X and Y commute: Like in [JP, Example 4.6], one sees that there is no $N \in \mathbb{N}$ such that $N - (X^2 + Y^2) \in \pi(M_S)$.

Unlike in the commutative case, we have to work a bit to show that K_S is bounded. Let $(A, B) \in K_S$ and set $C := A + B$, $D := A - B$. Then $C + D \geq 2$, $C - D \geq 2$ and $C^2 - D^2 \leq 16$. Considering the operator norms $c := \|C\|$ and $d := \|D\|$, we get $d \leq c - 2$ from $\pm D \leq C - 2$ and $c^2 \leq 16 + d^2$ from the corresponding fact with capital letters. This implies $c^2 \leq 16 + (c - 2)^2 = 16 + c^2 - 4c + 4$, i.e., $c \leq 5$ and $d \leq 3$. We finally obtain $2\|A^2 + B^2\| = \|C^2 + D^2\| \leq c^2 + d^2 \leq 34$, whence $17 - A^2 - B^2 \geq 0$.

In our terminology, the main result of [HM] characterizes symmetric polynomials (everywhere) positive semidefinite on (bounded) K_S with M_S Archimedean and can be stated as follows:

1.2. Theorem (Theorem 1.2 in [HM]): *Let $S \cup \{f\} \subseteq \operatorname{Sym} k\langle\bar{X}\rangle$ and suppose that M_S is Archimedean. If $f(A) > 0$ for all $A \in K_S$, then $f \in M_S$.*

Together with the following proposition whose proof is left to the reader, we see that under the assumptions of the preceding theorem, f is positive semidefinite on K_S if and only if $f + \varepsilon \in M_S$ for all $\varepsilon \in \mathbb{R}_{>0}$.

1.3. Proposition: *Let $S \cup \{f\} \subseteq \operatorname{Sym} k\langle\bar{X}\rangle$. If $f \in M_S$, then $f(A) \geq 0$ for all $A \in K_S$.*

Our main result characterizes symmetric polynomials that are *nowhere* negative semidefinite on K_S with M_S Archimedean. We are indebted to Prof. Dr. J. Cimprič who motivated us to look at this problem. After having completed this work, we have received his work [C2] containing a different and abstract approach to the same problem (see [C2, Theorem 5]).

1.4. Theorem (Nirgendsnegativsemidefinitheitsstellensatz): *The following conditions are equivalent for all $S \cup \{f\} \subseteq \operatorname{Sym} k\langle\bar{X}\rangle$ provided that M_S is Archimedean:*

(i) *$f(A) \not\leq 0$ for all $A \in K_S$.*

(ii) *There exist $r \in \mathbb{N}$ and $h_1, \ldots, h_r \in k\langle\bar{X}\rangle$ with $\sum_{i=1}^{r} h_i^* f h_i \in 1 + M_S$.*

As an application, we characterize polynomials that are positive semidefinite on the 'noncommutative cube' as those polynomials that can be approximated by sums of Hermitian squares. Here, the points of the 'noncommutative cube' are tuples of contractions. A **contraction** is a bounded operator A with operator norm $\|A\| \leq 1$. The approximation is with respect to the **1-norm** which we define for any polynomial $f = \sum_\alpha a_\alpha \bar{X}^\alpha$ in commuting or noncommuting variables with coefficients a_α as

$$\|f\|_1 := \sum_\alpha |a_\alpha|.$$

1.5. Theorem: *Suppose $f \in k\langle \bar{X} \rangle$ is a polynomial of degree d, and set $s := \sum_{i=0}^d n^i$. The following are equivalent:*

(i) *$f(A_1, \ldots, A_n)$ is positive semidefinite for all k-Hilbert spaces E and all contractive self-adjoint operators $A_1, \ldots, A_n$ on E.*

(ii) *$f(A_1, \ldots, A_n)$ is positive semidefinite for all contractive self-adjoint matrices $A_1, \ldots, A_n \in k^{s \times s}$.*

(iii) *$f \in \overline{M_\varnothing}$ with respect to the 1-norm, i.e., for all $\varepsilon \in \mathbb{R}_{>0}$, there exist $r \in \mathbb{N}$ and $g_1, \ldots, g_r \in k\langle \bar{X} \rangle$ such that*

$$\left\| f - \sum_{\ell=1}^r g_\ell^* g_\ell \right\|_1 < \varepsilon.$$

This result can be viewed as a noncommutative analog of the following well-known result of Berg, Christensen and Ressel.

1.6. Theorem (§9 in [BCR]): *For every polynomial $f \in \mathbb{R}[Y_1, \ldots, Y_n]$ in commuting variables Y_i, the following are equivalent:*

(i) *$f \geq 0$ on the cube $[-1, 1]^n$.*

(ii) *For all $\varepsilon \in \mathbb{R}_{>0}$, there exist $r \in \mathbb{N}$ and $g_1, \ldots, g_r \in \mathbb{R}[Y_1, \ldots, Y_n]$ such that $\|f - \sum_{\ell=1}^r g_\ell^* g_\ell\|_1 < \varepsilon$.*

2. Proofs

2.1. Definition: To every quadratic module $M \subseteq k\langle \bar{X} \rangle$ we associate its **ring of bounded elements**

$$H(M) := \{f \in k\langle \bar{X} \rangle \mid N - f^* f \in M \text{ for some } N \in \mathbb{N}\}.$$

This is indeed a ring, even a k-subalgebra of $k\langle \bar{X} \rangle$ as proved in [Vid, Lemma 4] (see also [C1, Section 2] and [S2, Section 2]).

2.2. Proposition: *A quadratic module $M \subseteq k\langle\bar{X}\rangle$ is Archimedean if and only if $H(M) = k\langle\bar{X}\rangle$.*

Proof: Suppose $N \in \mathbb{N}$ is such that

$$N - (X_1^2 + \cdots + X_n^2) \in M.$$

It follows that

$$N - X_i^2 = N - (X_1^2 + \cdots + X_n^2) + \sum_{j \neq i} X_j^2 \in M,$$

so $X_i \in H(M)$ for all i. Since $H(M)$ is a k-algebra, this implies that $H(M) = k\langle\bar{X}\rangle$. ∎

In the terminology of Köthe [Köt], this proposition together with the identity

$$s = \left(\frac{s+1}{2}\right)^2 - \left(\frac{s-1}{2}\right)^2$$

shows that M is an Archimedean quadratic module if and only if 1 is an algebraic interior point of the convex cone $M \subseteq \operatorname{Sym} k\langle\bar{X}\rangle$. Recall, $f \in M$ is an **algebraic interior point** if for every $p \in \operatorname{Sym} k\langle\bar{X}\rangle$ there exists $\varepsilon > 0$ with $f + \varepsilon p \in M$.

Proof of Theorem 1.4: The easy part is to show that (ii) implies (i) (for which the Archimedean property of M_S is not needed). Suppose $\sum_i h_i^* f h_i \in 1 + M_S$ for some $h_i \in k\langle\bar{X}\rangle$. By Proposition 1.3, $(h_i^* f h_i)(A) \geq 1$. For every nonzero vector $v \in E$,

$$\sum_i \langle f(A)h_i(A)v, h_i(A)v\rangle = \sum_i \langle h_i^*(A)f(A)h_i(A)v, v\rangle \geq \langle v, v\rangle > 0.$$

Hence (compare with the footnote on page 2) $f(A) \not\leq 0$.

The proof of the nontrivial part proceeds in several steps.

Step 1: Separation.
Let C denote the set of all (finite) sums of elements $h^* f h$ ($h \in k\langle\bar{X}\rangle$). This is a convex cone in the $\mathbb{R}$-vector space $\operatorname{Sym} k\langle\bar{X}\rangle$. We claim that $C \cap (1 + M_S) \neq \varnothing$. Assume otherwise. Then C contains no algebraic interior points of M_S. Thus we can apply Eidelheit's separation theorem [Köt, §17.1(3)]. There exists a nonzero $\mathbb{R}$-linear functional L_0: $\operatorname{Sym} k\langle\bar{X}\rangle \to \mathbb{R}$ with $L_0(M_S) \subseteq \mathbb{R}_{\geq 0}$ and $L_0(C) \subseteq \mathbb{R}_{\leq 0}$. We now extend L_0 to a k-linear functional L: $k\langle\bar{X}\rangle \to k$ satisfying

$$(1) \qquad L(M_S) \subseteq \mathbb{R}_{\geq 0} \quad \text{and} \quad L(C) \subseteq \mathbb{R}_{\leq 0}.$$

In case $k = \mathbb{R}$, we define $L(p) := L_0(p^* + p)$ for $p \in k\langle \bar{X} \rangle$. Now consider the case $k = \mathbb{C}$. The identity

$$p = \frac{p + p^*}{2} + i\frac{p - p^*}{2i}, \quad p \in k\langle \bar{X} \rangle$$

gives a direct sum decomposition of $\mathbb{C}\langle \bar{X} \rangle$ into two real vector spaces

$$\mathbb{C}\langle \bar{X} \rangle = \operatorname{Sym} \mathbb{C}\langle \bar{X} \rangle \oplus i \operatorname{Sym} \mathbb{C}\langle \bar{X} \rangle.$$

Now define L by

$$L(p + iq) := L_0(p) + iL_0(q), \quad p, q \in \operatorname{Sym} \mathbb{C}\langle \bar{X} \rangle.$$

It is easy to check that L is $\mathbb{C}$-linear.

Step 2: The Gelfand–Naimark–Segal construction.
By the Cauchy–Schwarz inequality for semi-scalar products,

$$N := \{p \in k\langle \bar{X} \rangle \mid L(p^* p) = 0\}$$

is a linear subspace of $k\langle \bar{X} \rangle$. Similarly, we see that

$$\langle \overline{p}, \overline{q} \rangle := L(q^* p) \tag{2}$$

defines a scalar product on $k\langle \bar{X} \rangle / N$, where $\overline{p} := p + N$ denotes the residue class of $p \in k\langle \bar{X} \rangle$ modulo N. Let E denote the completion of $k\langle \bar{X} \rangle / N$ with respect to this scalar product. Since $1 \notin N$ (otherwise, $L = 0$ because M_S is Archimedean), E is nontrivial.

Step 3: Construction of $\hat{X} \in K_S$.
To prove that N is a left ideal of $k\langle \bar{X} \rangle$, we fix $i \in \{1, \ldots, n\}$ and show that $X_i N \subseteq N$. Since M_S is Archimedean, there is some $m \in \mathbb{N}$ with $m - X_i^2 \in M_S$ for every i. Hence for all $p \in k\langle \bar{X} \rangle$, we have

$$0 \leq L(p^*(m - X_i^2)p) \leq mL(p^* p). \tag{3}$$

Now (3) shows that $L(p^* X_i^2 p) = 0$ for all $p \in N$, i.e., $X_i p \in N$.

Because N is a left ideal, the map

$$\Lambda_i \colon k\langle \bar{X} \rangle / N \to k\langle \bar{X} \rangle / N, \quad \overline{p} \mapsto \overline{X_i p}$$

is well-defined for each i. Obviously, it is linear and it is self-adjoint by the definition (2) of the scalar product. By (3), Λ_i is bounded and thus extends to a self-adjoint operator $\hat{X}_i$ on E. We claim that $\hat{X} := (\hat{X}_1, \ldots, \hat{X}_n) \in K_S$. For this, let $p \in S$ and $v \in E$ be arbitrary. Without loss of generality, $v = \overline{h}$ for some $h \in k\langle \bar{X} \rangle$. Hence,

$$\langle p(\hat{X})v, v\rangle = \langle \overline{ph}, \overline{h}\rangle = L(h^* ph) \geq 0,$$

because of $h^* ph \in M_S$ and (1).

Since f is nowhere negative semidefinite on K_S, there is some $v \in E$ with $\langle f(\hat{X})v, v\rangle > 0$. As before, we may assume that $v = \overline{h}$ for some $h \in k\langle \bar{X} \rangle$. From (1), we get

$$0 < \langle f(\hat{X})\overline{h}, \overline{h}\rangle = \langle \overline{fh}, \overline{h}\rangle = L(h^* fh) \leq 0,$$

a contradiction. ∎

We now turn to the proof of Theorem 1.5. The following proposition shows that all polynomial inequalities $f \geq 0$ holding for all self-adjoint (contraction) matrices are symmetric in the sense that $f = f^*$.

2.3. Proposition: *If $f \in k\langle \bar{X} \rangle$ of degree $< d$ satisfies $f(A_1, \ldots, A_n) = 0$ for all self-adjoint contractions $A_1, \ldots, A_n \in k^{d \times d}$, then $f = 0$.*

Proof: We form the $*$-polynomial $g := f((X_1 + X_1^*)/2, \ldots, (X_n + X_n^*)/2)$ in noncommuting variables $X_1, \ldots, X_n, X_1^*, \ldots, X_n^*$. It follows from the assumption that $g(A_1, \ldots, A_n) = 0$ for all contractions $A_1, \ldots, A_n \in k^{d \times d}$. Modify the multilinearization process explained in [Row] as follows: The formula for Δ_{iu} given in [Row, page 126] should be replaced by

$$\begin{aligned} \Delta_{iu} g = & g\big(X_1, \ldots, (X_i + X_u)/2, \ldots, X_n\big) - g\big(X_1, \ldots, X_i/2, \ldots, X_n\big) \\ & - g\big(X_1, \ldots, X_u/2, \ldots, X_n\big). \end{aligned}$$

Repeating this, we obtain a multilinear $*$-polynomial of degree $< d$ killing all $d \times d$ contractions. But this is impossible by the general theory of polynomial identities (cf. [Row, Remark 2.5.14]). ∎

The following lemma implies Theorem 1.5 in the very special case $f = 1 - X_i^2$, which could also be deduced from Theorem 1.6.

2.4. Lemma: *For all $m \in \mathbb{N}$, $1 - Y^2 + \frac{1}{m} Y^{2m}$ is a sum of squares in the polynomial ring $\mathbb{R}[Y]$.*

Proof: Check that

$$1 - Y^2 + \frac{1}{m} Y^{2m} = \frac{1}{m} + \frac{1}{m}(1 - Y^2)^2 \sum_{k=0}^{m-2} (m - 1 - k) Y^{2k}. \qquad ∎$$

Proof of Theorem 1.5: To see the easy implication from (iii) to (ii), let $A := (A_1, \dots, A_n)$ be a tuple of contractive self-adjoint $s \times s$ matrices. It is easy to see that (iii) implies $f = f^*$, in particular $f(A) = f(A)^*$. Fix a vector $v \in k^s$ with $\|v\| = 1$. We show that

$$\langle f(A)v, v\rangle > -\varepsilon$$

for all $\varepsilon \in \mathbb{R}_{>0}$. Choose $r \in \mathbb{N}$ and $g_1, \dots, g_r \in k\langle \bar{X}\rangle$ such that $\|h\|_1 < \varepsilon$ for $h := f - \sum_{\ell=1}^r g_\ell^* g_\ell$. Note that $h(A)$ is self-adjoint whence

$$\langle f(A)v, v\rangle = \langle h(A)v, v\rangle + \sum_{\ell=1}^{r} \langle g_\ell(A)v, g_\ell(A)v\rangle \geq -\|h(A)\| \geq -\|h\|_1 > -\varepsilon.$$

Let us now prove that (ii) implies (i) by contraposition. We assume that there exists a tuple $A := (A_1, \dots, A_n)$ of self-adjoint contractions on a k-Hilbert space E for which $f(A)$ is not positive semidefinite. Let $v \in E$ be a vector with $\langle f(A)v, v\rangle < 0$ and define

$$V := \operatorname{Span}\{A^w v \mid w \text{ is a word of length } \leq d\}.$$

V is a finite dimensional k-vector space with $\dim V \leq s$. Let $\pi \colon E \to V$ denote the orthogonal projection and define $B_i := \pi A_i \pi$. Obviously, B_i is a self-adjoint contraction and

$$\langle f(B_1, \dots, B_n)v, v\rangle = \langle f(A)v, v\rangle < 0.$$

It remains to show that (i) implies (iii). Assume that (i) holds. Since $f + \frac{1}{m}$ converges to f in the 1-norm when $m \to \infty$, we may assume that f is positive definite for all n-tuples of self-adjoint contractive operators. Let $S := \{1 - X_1^2, \dots, 1 - X_n^2\}$. By assumption, f is positive definite on K_S and M_S is Archimedean. Thus by Theorem 1.2, $f \in M_S$, i.e., f can be written as

$$f = \sum_i g_i^* g_i + \sum_{j=1}^{n} \sum_i g_{ij}^* (1 - X_j^2) g_{ij},$$

for some $g_i, g_{ij} \in k\langle \bar{X}\rangle$. Hence it suffices to show that $g^*(1 - X^2)g \in \overline{M_\varnothing}$ for every $g \in k\langle \bar{X}\rangle$. Identifying $\mathbb{R}[Y]$ from Lemma 2.4 with $\mathbb{R}[X_j] \subseteq k\langle \bar{X}\rangle$, we see that $1 - X_j^2 \in \overline{M_\varnothing}$. Noting that the map $k\langle \bar{X}\rangle \to k\langle \bar{X}\rangle, p \mapsto g^* p g$ is bounded with respect to the 1-norm (left or right multiplication by any variable is even an isometry), this finishes the proof. ∎

3. Concluding remarks

In Theorem 1.4, the hypothesis that M_S is Archimedean cannot be dropped. Otherwise boundedness of K_S would imply that M_S is Archimedean (contradicting Example 1.1). In the commutative case, the analogous implication holds often (see [Mar, JP]), for example, when S is a singleton (a quadratic module generated by one element is a preordering and for preorderings Schmüdgen's Theorem [S1, Corollary 3] holds). In the noncommutative case, the situation is different: In the next example, S is a singleton, $K_S = \varnothing$ and yet M_S is not Archimedean.

3.1. Example: Let $n = 2$, $k = \mathbb{C}$ and write $\bar{X} = (X, Y)$, i.e., we consider $k\langle \bar{X} \rangle = \mathbb{C}\langle X, Y \rangle$. Set

$$S := \{-(XY - YX + i)(XY - YX + i)^*\}.$$

We claim that $K_S = \varnothing$ but M_S is not Archimedean (though S is a singleton).

Assume that there exists $(A, B) \in K_S$. Then $AB - BA = -i$ and by induction $A^m B - BA^m = -imA^{m-1}$ for all $m \in \mathbb{N}$. Since A is self-adjoint, $\|A^m\| = \|A\|^m$ and therefore

$$m\|A\|^{m-1} = \|A^m B - BA^m\| \leq 2\|A\|^m \|B\|$$

for all $m \in \mathbb{N}$. Noting that $A \neq 0$, this yields that $m \leq 2\|A\|\|B\|$ for all $m \in \mathbb{N}$, a contradiction. Therefore $K_S = \varnothing$.

In order to show that M_S is not Archimedean, we use the Schrödinger representation (cf. [S2])

$$\begin{aligned} \pi_0\colon \mathbb{C}\langle X, Y \rangle &\to \mathcal{L}(S(\mathbb{R})) \\ X &\mapsto (f \mapsto -if') \\ Y &\mapsto (f \mapsto (x \mapsto xf(x))) \end{aligned}$$

where $S(\mathbb{R})$ is the dense subspace of $L^2(\mathbb{R})$ consisting of all Schwartz (i.e., 'rapidly decreasing smooth') functions $\mathbb{R} \to \mathbb{C}$, $\mathcal{L}(S(\mathbb{R}))$ is the vector space of all $\mathbb{C}$-linear operators $S(\mathbb{R}) \to S(\mathbb{R})$ and f' denotes the derivative of $f \in S(\mathbb{R})$. Note that π_0 is a $\mathbb{C}$-algebra homomorphism which respects the involution $*$ since $\pi_0(X)$ and $\pi_0(Y)$ are self-adjoint (for $\pi_0(X)$ this follows from integration by parts). Observing that $\pi_0(XY - YX + i) = 0$, we get that π_0 sends every element of M_S to a positive semidefinite operator on $S(\mathbb{R})$. Finally, $N - \pi_0(Y^2) \not\succeq 0$ regardless of $N \in \mathbb{N}$.

In spite of the last example, we do not know if the Nirgendsnegativsemidefinitheitsstellensatz 1.4 holds for $S = \varnothing$ (where M_S fails to be Archimedean).

3.2. Open Problem: *Regard the following conditions for a symmetric polynomial* $f \in k\langle \bar{X} \rangle$.

(i) $f(A_1, \ldots, A_n)$ *is not negative semidefinite for any* k*-Hilbert space* $E \neq \{0\}$ *and bounded self-adjoint operators* $A_1, \ldots, A_n$ *on* E.

(ii) *There are* $r \in \mathbb{N}$ *and* $g_1, \ldots, g_r \in k\langle \bar{X} \rangle$ *such that* $\sum_{i=1}^r g_i^* f g_i \in 1 + M_\varnothing$.

Obviously, (ii) *implies* (i). *Does* (i) *imply* (ii)*?*

An affirmative answer to this problem would give a noncommutative analog of Artin's solution to Hilbert's 17th problem (see for example [Rez] or [Mar]).

Acknowledgements: We thank Prof. Dr. K. Schmüdgen for bringing Eidelheit's separation theorem to our attention. Our sincere thanks also to Prof. Dr. J. Cimprič whose talk at the Luminy conference on Positive Polynomials brought up the idea of investigating nowhere negative semidefinite polynomials.

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